Some infinite classes of asymmetric nearly hamiltonian snarks

C. Fiori, B. Ruini *

Abstract

We determine the full automorphism group of each member of three infinite families of connected cubic graphs which are snarks. A graph is said to be nearly hamiltonian if it has a cycle which contains all vertices but one. We prove, in particular, that for every possible order $n \geq 28$ there exists a nearly hamiltonian snark of order n with trivial automorphism group.

Keywords: full automorphism group, snark, nearly hamiltonian snark. MSC(2000): 05C15, 20B25.

1 Introduction

Snarks are non-trivial connected cubic graphs which do not admit a 3-edge-coloring (a precise definition will be given below). The term snark owes its origin to Lewis Carroll's famouse nonsense poem "The Hunting of the Snark". It was introduced as a graph theoretical term by Gardner in [13] when snarks were thought to be very rare and unusual "creatures". Tait initiated the study of snarks in 1880 when he proved that the Four Color Theorem is equivalent to the statement that no snark is planar. *Asymmetric graphs* are graphs possessing a single graph automorphism -the identity- and for that reason they are also called identity graphs. Twenty-seven examples of asymmetric graphs are illustrated in [27]. Two of them are the snarks Sn8 and Sn9 of order 20 listed in [21] p. 276. Asymmetric graphs have been the subject of many studies, see, for example, [4], [9], and [17]. Erdős and Rényi proved in [9] that almost all graphs are asymmetric. This property remains true also for cubic graphs, see [3]. Determining the full automorphism group of a given graph may require some non-trivial work, especially if the graph belongs to an infinite family and the task is that of determining the automorphism group of each member of the family. In this paper we are interested in the computation of the full automorphism group of each

*Dipartimento di Matematica, Università di Modena e Reggio Emilia, via Campi 213/B, 41100 Modena (Italy)

member of three infinite classes of snarks. We prove, in particular, that for every possible order ≥ 28 there exists an asymmetric (nearly hamiltonian) snark of that order.

Throughout the paper, $G = (V(G), E(G))$ will be a finite connected simple graph with vertex-set $V(G)$ and edge-set $E(G)$. The *chromatic index* $\chi'(G)$ of a graph G is the minimum number of colors needed to color the edges of G in such a way that no two adjacent edges are assigned one and the same color. If $\Delta(G)$ denotes the maximum degree of G then, since edges sharing a vertex require different colors, we have $\chi'(G) \geq \Delta(G)$. Vizing [23] proved that $\Delta(G)+1$ colors suffice: if G is a simple connected graph with maximum degree $\Delta(G)$, then the chromatic index $\chi'(G)$ satisfies the inequalities $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$. This result divides simple graphs into two classes: a simple graph G is *Class 1* if $\chi'(G) = \Delta(G)$, otherwise G is *Class 2*. Erdős and Wilson in [10] proved that, almost all graphs are Class 1. A *snark* is a cubic graph Class 2 and girth ≥ 5 which is cyclically 4–edge–connected (see Section 2). Some authors use a slightly different notion of a snark (see, for examples, [18] and [22]). The importance of snarks partially arises from the fact that some conjectures about graphs would have snarks as minimal counter-examples, see for example [16]: (a) (Tutte's 5-Flow Conjecture) every bridgeless graph has a nowhere-zero 5-flow; (b) (The 1-Factor Double Cover Conjecture) every bridgeless cubic graph can be covered exactly twice with 1-factors; (c) (The Cycle Double Cover Conjecture) every bridgeless graph can be covered exactly twice with cycles.

The first graph which was shown to be a snark is the Petersen graph discovered in 1898. Up until 1975 only four examples of snarks were known. In 1975 Isaacs [15] produced the first two infinite families of snarks. In [1], [2] and [5] a catalogue of snarks of order smaller than 30 is generated. For survey papers on snarks we refer the reader to [6], [5], [8] and [24]. If a connected graph G admits a cycle containing all vertices but one, then we shall say that G is *nearly hamiltonian*. In the paper [6] graphs with this property were referred to as almost-hamiltonian graphs, but we prefer to avoid this terminology here because the term almost-hamiltonian has been used with different meanings elsewhere, see [19], [20], [25]. In this paper we are interested in nearly hamiltonian snarks. In [6] several classical snarks are shown to be nearly hamiltonian: the Celmins snark [7] of order 26, the Flower snark [15] of order 4k, the Double Star snark [12], the Goldberg snark [14] of order 8k, the Szekeres snark [24], the Watkins snarks [24] of order 42 and 50. Moreover, a catalogue of all non-isomorphic nearly hamiltonian snarks of order smaller than 30 is produced in [6]. In particular the following results holds [6, Thm. 1.1]: (a) All snarks of order less than 28 are nearly hamiltonian; (b) There are exactly 2897 non-isomorphic nearly hamiltonian snarks of order 28; (c) up to isomorphism, there is a unique nearly hamiltonian snark, of order 28 and girth ≥ 6, that is, the flower snark of order 28; and (d) there are exactly three snarks of order 28 which are not nearly hamiltonian.

Finally, in [6], a general method to construct some infinite families of nearly hamiltonian snarks is described. In Section 2 we recall this construction in detail and in Section 3 we focus our attention on three infinite families $\mathcal{O}, \mathcal{F}, \mathcal{I}$ obtained by applying the above mentioned construction. The first and the third family have been introduced in [6], while the second family is introduced here. In Section 4 we analyse the behavior of graph automorphisms on subgraphs which arise from the construction. In Section 5, by using the results of Section 4, we show that for every possible order $n \geq 28$ there exists an asymmetric nearly hamiltonian snark of order n belonging to the set $\mathcal{O} \cup \mathcal{F} \cup \mathcal{I}$.

2 Preliminaries

A path of length r in a graph G is a sequence of distinct edges of type $[v_0, v_1]$, $[v_1, v_2]$, $\ldots$, $[v_{r-1}, v_r]$. If all the vertices of the path are distinct, except for v_0 and v_r which coincide, then the path is a *cycle of length* r or r*-cycle* and we denote it as $(v_0, v_1, \ldots, v_r)$. The *girth* of G is the length of the shortest cycle of G. The graph G is *cyclically* k*-edge-connected* if deleting fewer than k edges from G does not disconnect G into components, each of which contains a cycle. According to our definition in Section 1 a *snark* is a cubic graph Class 2 and girth ≥ 5 which is cyclically 4–edge–connected.

We recall in detail a construction of infinite classes of nearly hamiltonian snarks described in [6]. Let H be the cubic graph of order 13, with five semi-edges $e_1, e_2, \ldots, e_5$, constructed as follows. Order the first twelve vertices in a circular way and assign a number to each one of them in the clockwise order, starting from the vertex 0. The thirteenth vertex is labelled 12. The edges of H are given by the pairs:

(a) [i,i+1] (indices mod 12), for any $i = 0, 1, \ldots, 11$, $i \neq 6, 9$;

(b) [3j,12], for any $j = 0, 1, 2$; and

(c) [1,5], [4,8], [7,10], and [9,11].

The remaining five edges $e_1, e_2, \ldots, e_5$, considered in the ordering induced by that of the vertices, are assumed to be semi-edges to make H a cubic graph. The girth of H is 5 (see Figure 1).

Starting from H we construct another cubic graph H^* of order 17, which has the same five semi-edges $e_1, e_2, \ldots, e_5$. We insert four new vertices, labelled a, b, c, and d, on the edges $[7, 8], [8, 9]$, and $[0, 11]$ of H so that the pairs $[7, a], [a, 8], [8, b], [b, 9], [11, c], [c, d], [d, 0], [a, c]$ and $[b, d]$ become edges of H^* (see Figure 1). The graph H^* has girth 5. The following result (see [6] p. 68) holds:

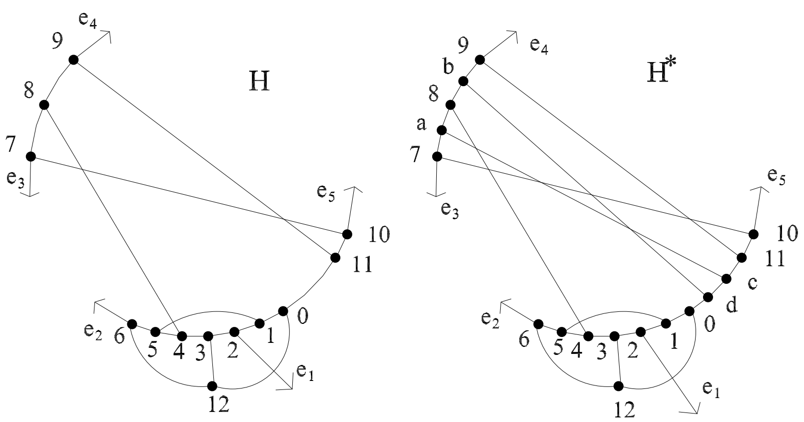

Figure 1: Graphs H and H^*

Theorem 2.1. *Let G be a snark of order n with a cutset of five edges whose removal leaves components H (as defined above) and F with semi-edges $\{e_1, e_2, e_3, e_4, e_5\}$ and $\{f_1, f_2, f_3, f_4, f_5\}$, respectively. Let G^* be the cubic graph obtained from G by replacing H with H^* and attaching the semi-edges of H^* to those of H according to the ordering induced by indices. Then G^* is a snark of order $n + 4$.*

The graph G^* contains H as a subgraph. Therefore we can repeat the construction an arbitrary number of times to obtain an infinite family of snarks. In particular we get the following corollary (see [6] p. 69):

Corollary 2.2. *Let G be a nearly hamiltonian snark of order n which contains H as subgraph. Let G_m be the snark obtained from G by applying the construction described in Theorem 2.1 m times. Then G_m is a nearly hamiltonian snark of order $n + 4m$.*

The infinite family $\{G_m\}_{m\geq 1}$ from Corollary 2.2 is said to be *generated by* the graph G.

By relabelling as in Figure 2 the vertices of the graph obtained from H^* by deleting the semi-edges we obtain the graph $G_X = (X, E(G_X))$, where $X = \{x_1, x_2, \ldots, x_{17}\}$ and $E(G_X) = \{[x_i, x_{i+1}] : i = 1, 2, \ldots, 16\} \cup \{[x_{10}, x_{17}], [x_{13}, x_{17}], [x_{11}, x_{15}], [x_3, x_{14}], [x_2, x_9], [x_1, x_7], [x_4, x_8]\}$ (the vertex x_{17} of G_X corresponds to the vertex 12 of H^*).

Consider the sets $A_m = \{a_i : i = 1, 2, \ldots, m-1\}$, $B_m = \{b_i : i = 1, 2, \ldots, m-1\}$, $C_m = \{c_i : i = 1, 2, \ldots, m-1\}$, $D_m = \{d_i : i = 1, 2, \ldots, m-1\}$ and let $\{\bar{G}_m\}_{m\geq 1}$ be the family of the graphs defined as follows:

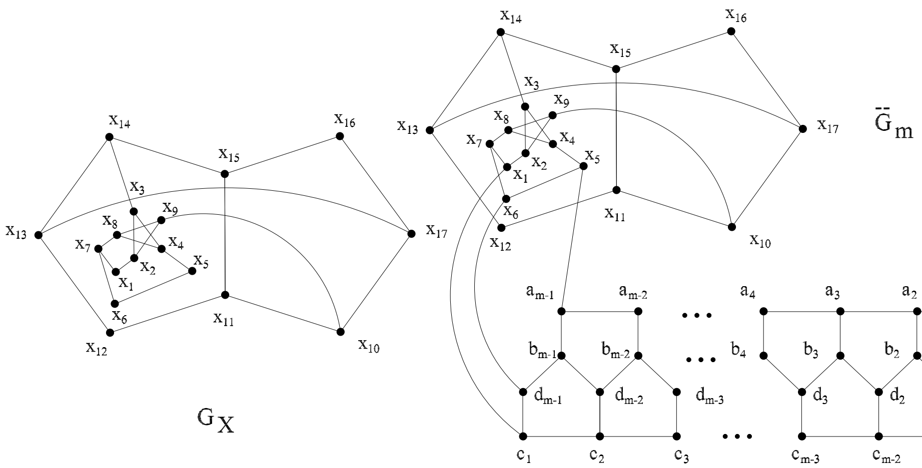

Figure 2: Graph $\bar{G}_m$ with $m \geq 1$

(1) $\bar{G}_m = G_X$ if $m = 1$;

(2) $\bar{G}_m = (V(\bar{G}_m), E(\bar{G}_m))$ if $m \geq 2$ with $V(\bar{G}_m) = X \cup A_m \cup B_m \cup C_m \cup D_m$ and $E(\bar{G}_m) = E(G_X) \cup \{[a_i, b_i] : i = 1, 2, \ldots, m-1\} \cup \{[b_i, d_i] : i = 1, 2, \ldots, m-1\} \cup \{[c_i, d_{m-i}] : i = 1, 2, \ldots, m-1\} \cup \{[x_5, a_{m-1}], [x_1, c_1], [x_6, d_{m-1}]\} \cup \cup \{[a_i, a_{i+1}] : i = 1, 2, \ldots, m-2\} \cup \{[c_i, c_{i+1}] : i = 1, 2, \ldots, m-2\} \cup \{[b_{i+1}, d_i] : i = 1, 2, \ldots, m-2\}$ where the last three sets are empty if $m = 2$.

The graph G_X is a subgraph of $\bar{G}_m$ for any $m \geq 1$ and the graph $\bar{G}_m$ is a subgraph of G_m (see Corollary 2.2) for any $m \geq 1$. Figure 2 illustrates the construction of the graph $\bar{G}_m$ with $m \geq 1$.

3 Three families $\mathcal{O}, \mathcal{F}, \mathcal{I}$ of nearly hamiltonian snarks

In this section we apply Corollary 2.2 to construct three infinite families of nearly hamiltonian snarks. Let O and F be the nearly hamiltonian snarks of order 24 shown in Figure 3, and let I be the nearly hamiltonian snark of order 26 shown in Figure 4. Dotted lines identify the cycle missing one vertex. By applying Corollary 2.2 to O, F and I we obtain three infinite families of nearly hamiltonian snarks $\mathcal{O} = \{O_m : m \geq 1\}$, $\mathcal{F} = \{F_m : m \geq 1\}$ and $\mathcal{I} = \{I_m : m \geq 1\}$ (generated by the snarks O, F and I, respectively).

The first and the third family have been introduced in [6] while the second family is new. Now we give an explicit description of the classes $\mathcal{O}$,

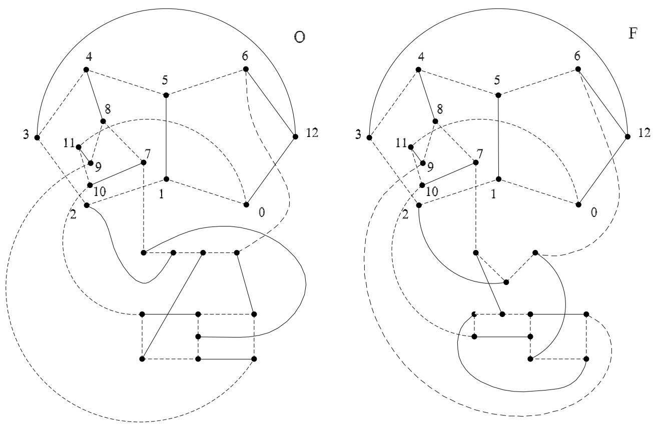

Figure 3: Nearly hamiltonian snarks O and F

$\mathcal{F}, \mathcal{I}$. Let us consider the following sets: $Y = \{y_1, y_2, y_3, y_4\}$, $T = \{t_1, t_2, t_3\}$, $Z = \{z_1, z_2, \ldots, z_7\}$, $S = \{s_1, s_2, s_3, s_4, s_5\}$, $R = \{r_1, r_2, \ldots, r_8\}$. Figures 4 and 5 show the graphs O_1, F_1 I_1. Also in this cases dotted lines identify the cycle missing one vertex. For $m \geq 2$ we get

- $O_m = (V(O_m), E(O_m))$ with $V(O_m) = V(\bar{G}_m) \cup Y \cup Z$ and $E(O_m) = E(\bar{G}_m) \cup \{[y_i, y_{i+1}] : i = 1, 2, 3\} \cup \{[z_i, z_{i+1}] : i = 1, 2, \ldots, 6\} \cup \{[x_{16}, y_1], [y_1, z_2], [y_2, z_6], [y_3, x_{12}], [y_4, a_1], [y_4, z_4], [z_1, z_5], [z_3, z_7], [z_1, c_{m-1}], [z_7, b_1]\}$;

- $F_m = (V(F_m), E(F_m))$ with $V(F_m) = V(\bar{G}_m) \cup T \cup R$ and $E(F_m) = E(\bar{G}_m) \cup \{[t_i, t_{i+1}] : i = 1, 2\} \cup \{[r_i, r_{i+1}] : i = 1, 2, \ldots, 7\} \cup \{[x_{16}, t_1], [x_{12}, t_2], [t_1, r_3], [t_3, r_6], [t_3, a_1], [r_2, r_7], [r_1, r_5], [r_4, r_8], [r_1, c_{m-1}], [r_8, b_1]\}$.

- $I_m = (V(I_m), E(I_m))$ with $V(I_m) = V(\bar{G}_m) \cup S \cup R$ and $E(I_m) = E(\bar{G}_m) \cup \{[s_i, s_{i+1}] : i = 1, 2, 3, 4\} \cup \{[r_i, r_{i+1}] : i = 1, 2, \ldots, 7\} \cup \{[x_{16}, s_1], [x_{12}, s_3], [s_1, r_2], [s_2, r_6], [s_4, r_4], [s_5, r_8], [s_5, c_{m-1}], [r_1, r_5], [r_3, r_7], [r_1, a_1], [r_8, b_1]\}$.

The graph $\bar{G}_m$ is a subgraph of each of O_m, F_m and I_m, with $m \geq 1$ and in particular the graph G_X is a subgraph of each $G \in \mathcal{O} \cup \mathcal{F} \cup \mathcal{I}$. Figure 6, 7 and 8 show the graphs O_m, F_m, I_m with $m = 4$. Also in these cases dotted lines identify the cycle missing one vertex.

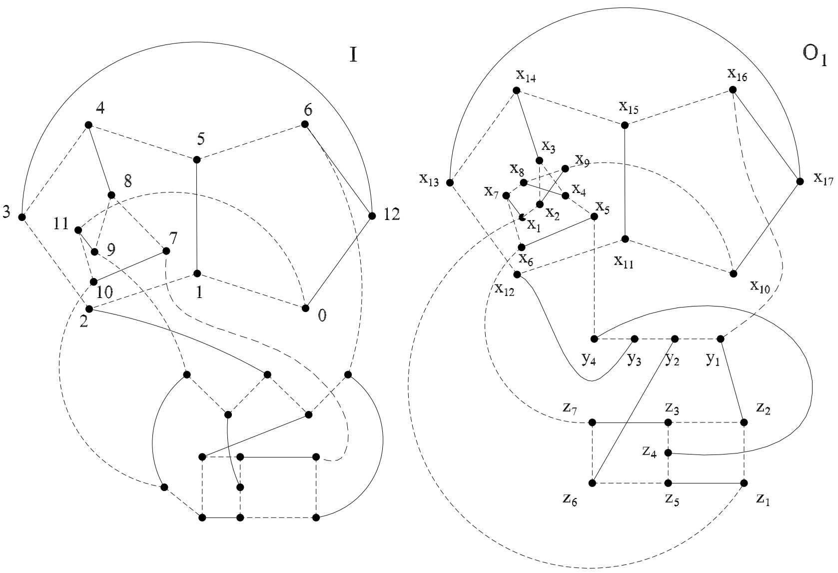

Figure 4: Nearly hamiltonian snarks I and O_1

4 Automorphisms of cubic graphs with $\bar{G}_m$ as a subgraph

In this section we describe the behavior of particular graph automorphisms of cubic graphs which act on the subgraphs $\bar{G}_m$. This will be useful in Section 5 for the computation of the full automorphism group of each graph of $\mathcal{O} \cup \mathcal{F} \cup \mathcal{I}$. We will be using the functional notation for mappings, in other words $\alpha(x)$ denotes the image of the element x under mapping α and $\alpha_{|A}$ denotes the restriction of α to A.

In what follows we shall make repeated use of the following four elementary properties of an automorphism of a cubic graph with girth at least five.

Elementary Properties *Let G be a cubic graph with girth at least 5 and let α be an automorphism of G. Then,*

$EP1$) the number of r-cycles passing through a vertex u of G coincides with the number of r-cycles passing through the vertex $\alpha(u)$;

$EP2$) if u and v are vertices fixed by α with the property of being adjacent to a vertex w, then the vertex w is also fixed by α;

$EP3$) if α fixes the vertices u, v, w with $[u, v], [w, v]$ and $[v, t]$ edges of G, then the vertex t is fixed by α;

$EP4$) if α fixes the vertices v, t and if u, w and t are different vertices adjacent to the vertex v, then $\alpha(\{u, w\}) = \{u, w\}$.

We note that EP2 follows from the observation that if the vertex w is not

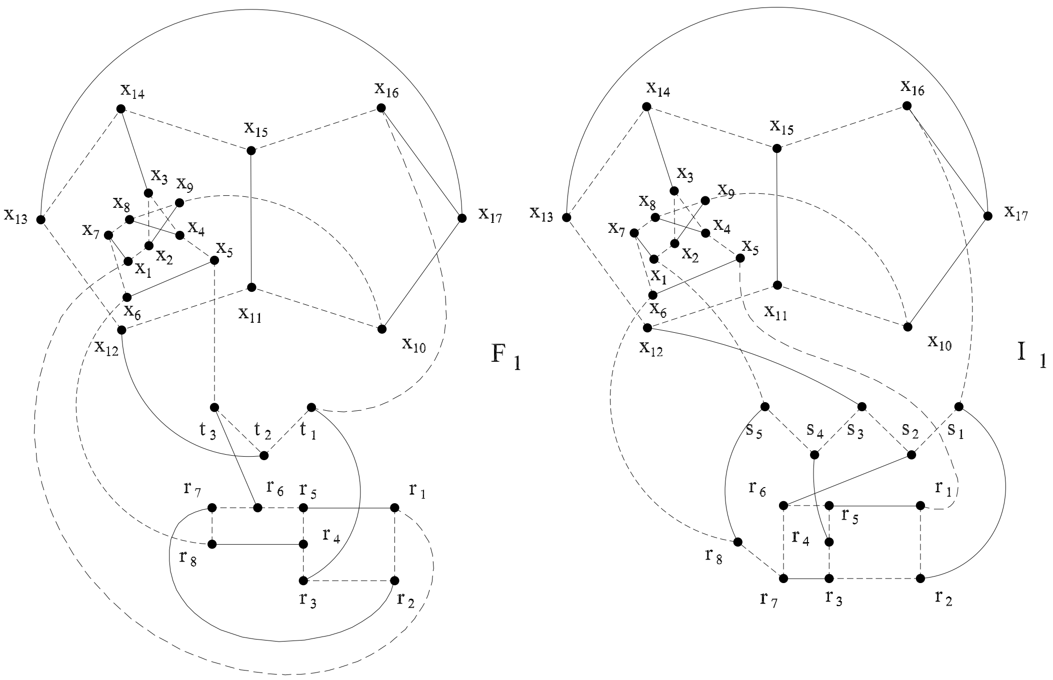

Figure 5: Nearly hamiltonian snarks F_1 and I_1

fixed by α then $(u, w, v, \alpha(w))$ would be a 4-cycle, contradicting the fact that the girth of G is at least 5.

Lemma 4.1. *Let G be a cubic graph with girth at least 5 and with G_X as a subgraph. The 5-cycles constituted by vertices of X are the following:*

$\mathcal{C}_1 = (x_{10}, x_{11}, x_{12}, x_{13}, x_{17})$, $\quad \mathcal{C}_2 = (x_{10}, x_{11}, x_{15}, x_{16}, x_{17})$,

$\mathcal{C}_3 = (x_{11}, x_{12}, x_{13}, x_{14}, x_{15})$, $\quad \mathcal{C}_4 = (x_{13}, x_{14}, x_{15}, x_{16}, x_{17})$,

$\mathcal{C}_5 = (x_4, x_5, x_6, x_7, x_8)$, $\quad \mathcal{C}_6 = (x_2, x_3, x_4, x_8, x_9)$,

$\mathcal{C}_7 = (x_1, x_2, x_9, x_8, x_7)$.

Moreover, the following are the only other 5-cycles that can pass through at least one vertex of X:

$\mathcal{C}_8 = (x_5, x_6, p_1, p_2, p_3)$, $\mathcal{C}_9 = (x_1, x_7, x_6, q_1, q_2)$ *with* $p_i, q_j \notin X$, $i = 1, 2, 3$, $j = 1, 2$,

$\mathcal{C}_{10} = (x_1, x_7, x_6, x_5, p)$, $\mathcal{C}_{11} = (x_{16}, r, x_{12}, x_{11}, x_{15})$, $\mathcal{C}_{12} = (x_{16}, r, x_{12}, x_{13}, x_{17})$ *with* $p, r \notin X$.

Proposition 4.2. *Let G be a cubic graph with girth at least 5 and with G_X as a subgraph. Let α be an automorphism of G with $\alpha(x_3) \in X$, then $\alpha(x_3) = x_1$ or $\alpha(x_3) = x_3$.*

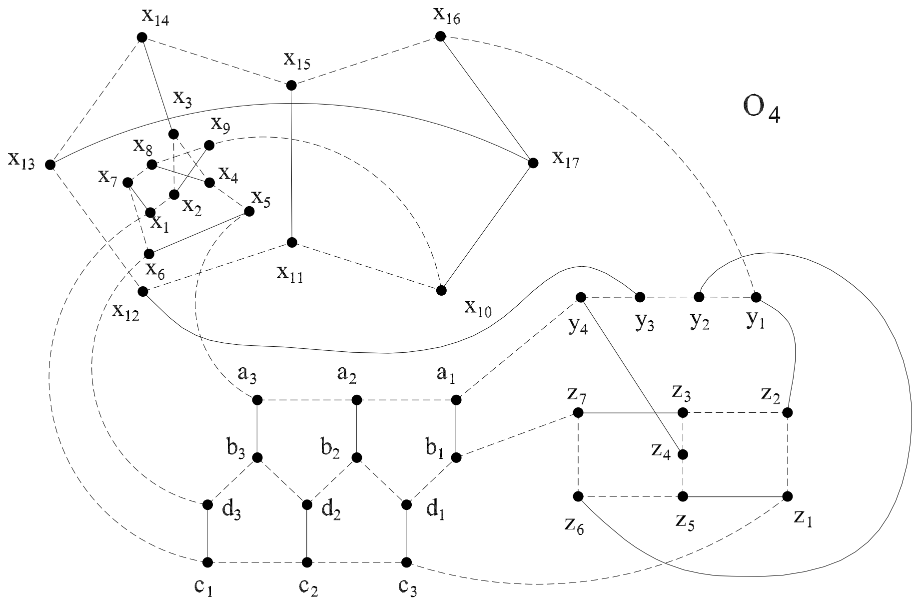

Figure 6: Nearly hamiltonian snark O_4

Proof. Lemma 4.1 establishes that $\mathcal{C}_6$ is the only 5-cycle in G passing through the vertex x_3 and that there are at least two 5-cycles in G going through each vertex of the set $X \setminus \{x_1, x_3, x_5, x_6\}$. Therefore, by Property EP1 we get that $\alpha(x_3) \in \{x_1, x_3, x_5, x_6\}$. If $\alpha(x_3) = x_5$ or $\alpha(x_3) = x_6$, then the 5-cycle $\mathcal{C}_5$ has to be the only 5-cycle in G touching x_5 or x_6. Hence the relation $\alpha(\mathcal{C}_6) = \mathcal{C}_5$ yields one of the following cases:

$$(a) \begin{cases} \alpha(x_2) = x_4 \\ \alpha(x_3) = x_5 \\ \alpha(x_4) = x_6 \\ \alpha(x_8) = x_7 \\ \alpha(x_9) = x_8 \end{cases} \quad (b) \begin{cases} \alpha(x_2) = x_6 \\ \alpha(x_3) = x_5 \\ \alpha(x_4) = x_4 \\ \alpha(x_8) = x_8 \\ \alpha(x_9) = x_7 \end{cases} \quad (c) \begin{cases} \alpha(x_2) = x_5 \\ \alpha(x_3) = x_6 \\ \alpha(x_4) = x_7 \\ \alpha(x_8) = x_8 \\ \alpha(x_9) = x_4 \end{cases} \quad (d) \begin{cases} \alpha(x_2) = x_7 \\ \alpha(x_3) = x_6 \\ \alpha(x_4) = x_5 \\ \alpha(x_8) = x_4 \\ \alpha(x_9) = x_8. \end{cases}$$

Cases (a) and (d) cannot occur since only two 5-cycles pass through the vertex x_9 while three 5-cycles go through the vertex x_8; thus $\alpha(x_9) \neq x_8$.

Case (b) implies $\alpha(\mathcal{C}_7) = \alpha((x_1, x_2, x_9, x_8, x_7)) = (\alpha(x_1), \alpha(x_2), \alpha(x_9), \alpha(x_8), \alpha(x_7)) = (\alpha(x_1), x_6, x_7, x_8, \alpha(x_7))$. Lemma 4.1 establishes that $\mathcal{C}_5 = (x_5, x_6, x_7, x_8, x_4)$ is the only 5-cycle touching the three vertices x_6, x_7, x_8, whereby $\alpha(x_7) = x_4$; a contradiction, since from (b) we have $\alpha(x_4) = x_4$.

Case (c) does not occur either. In G two 5-cycles pass through x_2 and $\alpha(x_2) = x_5$. Thus, there must be two 5-cycles going through the vertex x_5, either $\mathcal{C}_5$ and $\mathcal{C}_8$ or $\mathcal{C}_5$ and $\mathcal{C}_{10}$ (see Lemma 4.1). Therefore two 5-cycles touch the vertex x_6 (either $\mathcal{C}_5$ and $\mathcal{C}_8$ or $\mathcal{C}_5$ and $\mathcal{C}_{10}$), whereas only one 5-cycle pass through the vertex x_3. Thus $\alpha(x_3) = x_6$ is a contradiction.

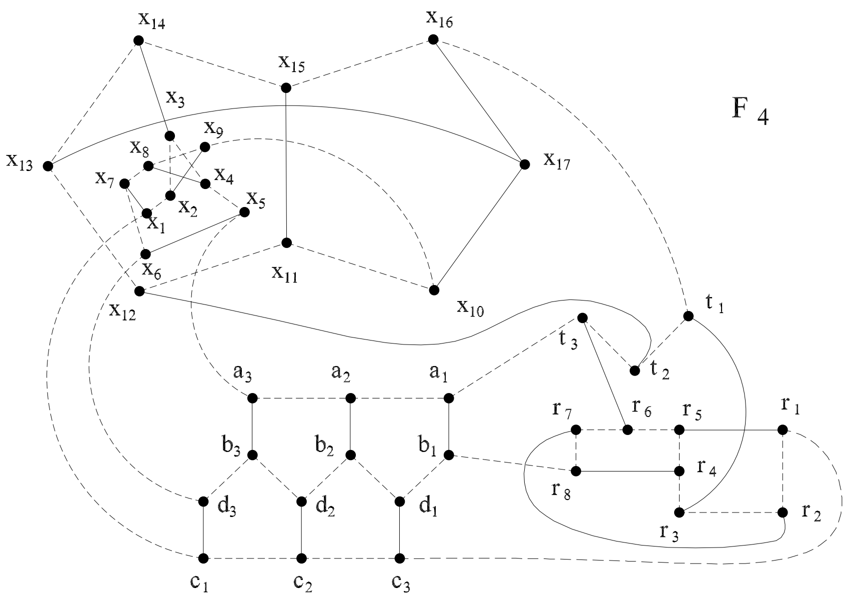

Figure 7: Nearly hamiltonian snark F_4

Therefore, if $\alpha(x_3) \in X$, then we have either $\alpha(x_3) = x_1$ or $\alpha(x_3) = x_3$. $\square$

Proposition 4.3. *Let G be a cubic graph with girth at least 5 and with G_X as a subgraph. Let α be an automorphism of G that fixes x_3. Then α fixes X setwise and the restriction $\alpha|X$ is either the identity permutation or the involution $(x_{11}\ x_{17})(x_{12}\ x_{16})(x_{13}\ x_{15})$.*

Proof. Let $\alpha(x_3) = x_3$ be. By Property EP1 and Lemma 4.1 we get $\alpha(\mathcal{C}_6) = \mathcal{C}_6$, hence $\alpha(x_8) \in \{x_8, x_9\}$; therefore, by Property EP1 and Lemma 4.1 we have $\alpha(x_8) = x_8$. From $\alpha(\mathcal{C}_6) = \mathcal{C}_6$, $\alpha(x_3) = x_3$, $\alpha(x_8) = x_8$ we obtain $\alpha(x_2) = x_2, \alpha(x_4) = x_4, \alpha(x_9) = x_9$. Property EP3 implies $\alpha(x_{10}) = x_{10}$, $\alpha(x_{14}) = x_{14}, \alpha(x_7) = x_7$. By Lemma 4.1 and $\alpha(x_4) = x_4, \alpha(x_8) = x_8$, $\alpha(x_7) = x_7$ we have $\alpha(\mathcal{C}_5) = \mathcal{C}_5$, thus $\alpha(x_6) = x_6, \alpha(x_5) = x_5$ and by Property EP2 we finally get $\alpha(x_1) = x_1$.

Since the automorphism α fixes x_3 and x_{14}, then Property EP4 implies that $\alpha(\{x_{13}, x_{15}\}) = \{x_{13}, x_{15}\}$.

Case I: if $\alpha(x_{13}) = x_{13}$ and $\alpha(x_{15}) = x_{15}$ then by Property EP2 we get $\alpha(x_{11}) = x_{11}$, $\alpha(x_{12}) = x_{12}$, $\alpha(x_{17}) = x_{17}$ and $\alpha(x_{16}) = x_{16}$. Therefore $\alpha(X) = X$ and the restriction of α to X is the identity permutation.

Case II: if $\alpha(x_{13}) = x_{15}$ and $\alpha(x_{15}) = x_{13}$ then $\alpha(\mathcal{C}_2) = \alpha((x_{10}, x_{17}, x_{16}, x_{15}, x_{11})) = (x_{10}, \alpha(x_{17}), \alpha(x_{16}), x_{13}, \alpha(x_{11}))$, hence $\alpha(x_{11})$ is adjacent to the vertices x_{13} and x_{10}, and so $\alpha(x_{11}) = x_{17}$. Moreover, $\alpha(\mathcal{C}_3) = \alpha((x_{11}, x_{12}, x_{13}, x_{14}, x_{15})) = (x_{17}, \alpha(x_{12}), x_{15}, x_{14}, x_{13})$, hence $\alpha(x_{12})$ is adjacent to the vertices x_{15}, x_{17}, and $\alpha(x_{12}) = x_{16}$; therefore $\alpha(x_{17}) = x_{11}$ and $\alpha(x_{16}) = x_{12}$. We have proved that $\alpha(X) = X$ and $\alpha|X = (x_{11}\ x_{17})(x_{12}\ x_{16})(x_{13}\ x_{15})$. $\square$

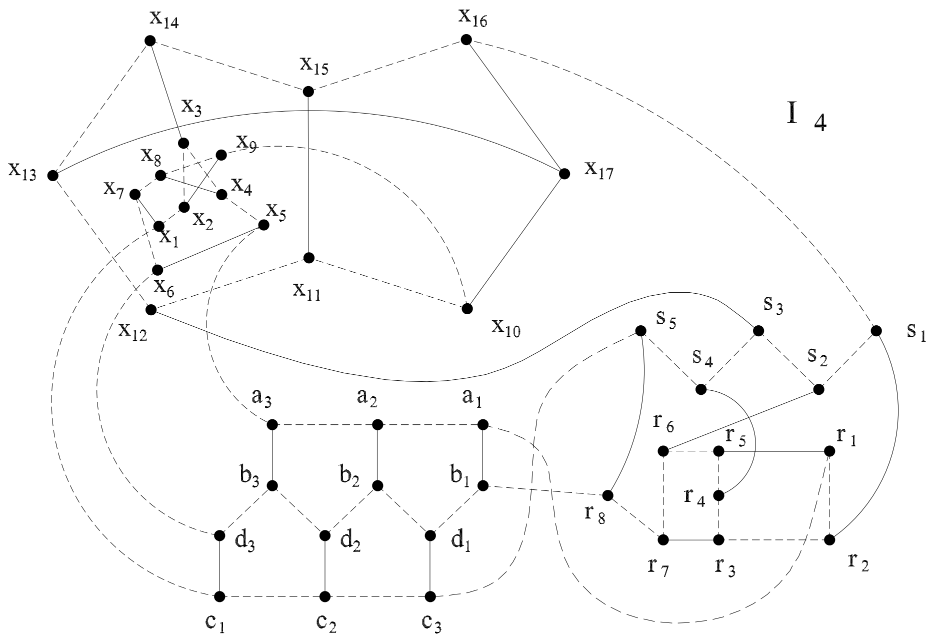

Figure 8: Nearly hamiltonian snark I_4

Proposition 4.4. *Let G be a cubic graph with girth at least 5 and with $\bar{G}_m$ as a subgraph for some $m \geq 2$. Let α be an automorphism of G that fixes x_3. Then α fixes $A_m \cup B_m \cup C_m \cup D_m$ pointwise.*

Proof. Let $\alpha(x_3) = x_3$. By Proposition 4.3 the vertex x_i, with $i = 1, 2, 4, 5, 6, 7$, is fixed by α and by Property EP3 we have $\alpha(c_1) = c_1$, $\alpha(a_{m-1}) = a_{m-1}$ and $\alpha(d_{m-1}) = d_{m-1}$; hence by Property EP3 we also get $\alpha(b_{m-1}) = b_{m-1}$. If $m = 2$ the statement is proved.

If $m \geq 3$ we prove
1) $\alpha(a_i) = a_i$ with $i = m-2, m-3, \ldots, 1$;
2) $\alpha(b_i) = b_i$ with $i = m-2, m-3, \ldots, 1$;
3) $\alpha(c_j) = c_j$ with $j = 2, 3, \ldots, m-1$;
4) $\alpha(d_i) = d_i$ with $i = m-2, m-3, \ldots, 1$.

The vertices a_{m-1}, x_5, b_{m-1} are fixed by α, hence Property EP3 implies that $\alpha(a_{m-2}) = a_{m-2}$. The vertices x_1, c_1, d_{m-1} are fixed by α, thus by Property EP3 we obtain $\alpha(c_2) = c_2$. The vertex d_{m-2} is adjacent to the fixed vertices c_2 and b_{m-1}, thus by Property EP2 we get $\alpha(d_{m-2}) = d_{m-2}$. The vertex b_{m-2} is adjacent to the fixed vertices a_{m-2} and d_{m-2}, so Property EP2 yields $\alpha(b_{m-2}) = b_{m-2}$. Let h be an integer $h \geq 2$. By induction we assume that 1), 2), 4) are true for $i \geq m-h$ and that 3) is true for $j \leq h$. We prove 1), 2), 4) for $i = m-(h+1)$ and 3) for $j = h+1$. By induction hypothesis the vertices b_{m-h}, a_{m-h} and $a_{m-(h-1)}$ are fixed by α, hence Property EP3 implies that $\alpha(a_{m-(h+1)}) = a_{m-(h+1)}$. The vertices c_h, c_{h-1} and d_{m-h} are fixed by α, thus by Property EP3 we obtain $\alpha(c_{h+1}) = c_{h+1}$.

The vertex $d_{m-(h+1)}$ is adjacent to the fixed vertices c_{h+1} and b_{m-h}, hence Property EP2 implies that $\alpha(d_{m-(h+1)}) = d_{m-(h+1)}$. The vertex $b_{m-(h+1)}$ is adjacent to the fixed vertices $a_{m-(h+1)}$ and $d_{m-(h+1)}$ and so by Property EP2 we get $\alpha(b_{m-(h+1)}) = b_{m-(h+1)}$. Therefore, the automorphism α fixes the vertices of the set $A_m \cup B_m \cup C_m \cup D_m$. $\square$

Corollary 4.5. *Let G be a cubic graph with girth at least 5 and with $\bar{G}_m$ as a subgraph for some $m \geq 1$. Let α be an automorphism of G that fixes x_3. Then α leaves $\bar{G}_m$ invariant and the restriction of α to $\bar{G}_m$ is either the involution $(x_{11}\ x_{17})\ (x_{12}\ x_{16})(x_{13}\ x_{15})$ or the identity permutation.*

Proof. The statement follows from Propositions 4.3 and 4.4. $\square$

5 $\mathcal{O}, \mathcal{F}, \mathcal{I}$: asymmetric nearly hamiltonian snarks

In this section we prove that each member of $\mathcal{O} \cup \mathcal{F} \cup \mathcal{I}$ is an asymmetric graph. First of all, we consider some properties of cycles of graphs from $\mathcal{O} \cup \mathcal{F} \cup \mathcal{I}$ which will be useful for characterizing the automorphisms of the graphs.

Lemma 5.1. *The vertices of O_m lying in just one 5-cycle are the following:*

(a) $x_1, x_3, x_5, x_6, z_1, z_2, z_6, z_7$ *if* $m = 1$;

(b) $x_3, a_1, b_1, z_1, z_2, z_6, z_7, c_{m-1}$ *if* $m \geq 2$.

If $m = 1$, there are exactly six 8-cycles going through the vertex x_3 while there are exactly three 8-cycles touching the vertex x_1.

Proof. The statement follows from the definition of O_m. The following two tables show the 5-cycles and the 8-cycles passing through each vertex considered in the statement:

	x_1	x_5, x_6	x_3	z_1, z_2	z_6, z_7	a_1, b_1	a_1, b_1	c_1	c_{m-1}
cycle of length 5	$\mathcal{C}_7$ if $m = 1$	$\mathcal{C}_5$ if $m = 1$	$\mathcal{C}_6$ if $m \geq 1$	$\mathcal{C}_{16}$ if $m \geq 1$	$\mathcal{C}_{17}$ if $m \geq 1$	$\mathcal{C}_{15}$ if $m = 2$	$\mathcal{C}_{13}$ if $m \geq 3$	$\mathcal{C}_{18}$ if $m = 2$	$\mathcal{C}_{14}$ if $m \geq 3$

	x_1	x_3
cycle of length 8	$\Omega_1, \Omega_2, \Omega_3$ if $m = 1$	$\Omega_4, \Omega_5, \Omega_6, \Omega_7, \Omega_{12}, \Omega_{13}$ if $m = 1$

where $\mathcal{C}_5$, $\mathcal{C}_6$, $\mathcal{C}_7$ are the cycles of Lemma 4.1, $\mathcal{C}_{13} = (a_1, a_2, b_2, d_1, b_1)$, $\mathcal{C}_{14} = (c_{m-2}, c_{m-1}, d_1, b_2, d_2)$, $\mathcal{C}_{15} = (a_1, x_5, x_6, d_1, b_1)$, $\mathcal{C}_{16} = (z_1, z_2, z_3, z_4, z_5)$, $\mathcal{C}_{17} = (z_6, z_7, z_3, z_4, z_5))$, $\mathcal{C}_{18} = (x_1, c_1, d_1, x_6, x_7)$, $\Omega_1 = (x_5, x_4, x_8, x_9, x_2, x_1, x_7, x_6)$, $\Omega_2 = (y_4, x_5, x_6, x_7, x_1, z_1, z_5, z_4)$, $\Omega_3 = (x_1, z_1, z_5, z_4, z_3, z_7, x_6, x_7)$, $\Omega_4 = (x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_3, x_2, x_9)$, $\Omega_5 = (x_{10}, x_9, x_8, x_4, x_3, x_{14}, x_{13}, x_{17})$, $\Omega_6 = (x_{10}, x_{11}, x_{15}, x_{14}, x_3, x_4, x_8, x_9)$, $\Omega_7 = (x_{10}, x_9, x_2, x_3, x_{14}, x_{15}, x_{16}, x_{17})$, $\Omega_{12} = (x_{12}, x_{13}, x_{14}, x_3, x_4, x_5, y_4, y_3)$, $\Omega_{13} = (x_5, x_4, x_3, x_2, x_9, x_8, x_7, x_6)$. $\square$

Lemma 5.2. *The vertices of F_m lying in just one 5-cycle are the following:*

(a) x_1, x_3, x_5, x_6 *if* $m = 1$*;*

(b) x_3, a_1, b_1, c_{m-1} *if* $m \geq 2$.

If $m = 1$ *there are exactly four* 8*-cycles going through the vertex* x_1 *while there are exactly six* 8*-cycles touching the vertex* x_3.

Proof. The statement follows from the definition of F_m. The following two tables show the 5-cycles and the 8-cycles passing through each vertex considered in the statement:

	x_1	x_5, x_6	x_3	a_1, b_1	a_1, b_1	c_1	c_{m-1}
cycle of length 5	$\mathcal{C}_7$ if $m = 1$	$\mathcal{C}_5$ if $m = 1$	$\mathcal{C}_6$ if $m \geq 1$	$\mathcal{C}_{15}$ if $m = 2$	$\mathcal{C}_{13}$ if $m \geq 3$	$\mathcal{C}_{18}$ if $m = 2$	$\mathcal{C}_{14}$ if $m \geq 3$

	x_1	x_3
cycle of length 8	$\Omega_1, \Omega_8, \Omega_9, \Omega_{10}$ if $m = 1$	$\Omega_4, \Omega_5, \Omega_6, \Omega_7, \Omega_{11}, \Omega_{13}$ if $m = 1$

where $\mathcal{C}_5$, $\mathcal{C}_6$, $\mathcal{C}_7$ are the cycles of Lemma 4.1, $\mathcal{C}_{13}$, $\mathcal{C}_{14}$, $\mathcal{C}_{15}$, $\mathcal{C}_{18}$, Ω_1, Ω_4, Ω_5, Ω_6, Ω_7, Ω_{13} are cycles of Lemma 5.1; $\Omega_8 = (x_1, r_1, r_2, r_3, r_4, r_8, x_6, x_7)$, $\Omega_9 = (t_3, x_5, x_6, x_7, x_1, r_1, r_5, r_6)$, $\Omega_{10} = (x_1, r_1, r_5, r_6, r_7, r_8, x_6, x_7)$, $\Omega_{11} = (x_{12}, x_{13}, x_{14}, x_3, x_4, x_5, t_3, t_2)$. $\square$

Lemma 5.3. *The vertices of* I_m *lying in just one* 5*-cycle are the following:*

(a) $s_5, x_3, x_5, r_1, r_2, r_6, r_7, r_8$ *if* $m = 1$*;*

(b) $s_5, x_3, a_1, r_1, r_2, r_6, r_7, r_8$ *if* $m \geq 2$.

Proof. The statement follows from the definition of I_m. The following table shows the 5-cycles going through each vertex considered in the statement:

	x_5	x_3	a_1	a_1	s_5, r_8	s_5, r_8	r_1, r_2	r_6, r_7
cycle of length 5	$\mathcal{C}_5$ if $m = 1$	$\mathcal{C}_6$ if $m \geq 1$	$\mathcal{C}_{15}$ if $m = 2$	$\mathcal{C}_{13}$ if $m \geq 3$	$\mathcal{C}_{19}$ if $m = 1$	$\mathcal{C}_{20}$ if $m \geq 2$	$\mathcal{C}_{21}$ if $m \geq 1$	$\mathcal{C}_{22}$ if $m \geq 1$

where $\mathcal{C}_5$, $\mathcal{C}_6$ are the cycles of Lemma 4.1, $\mathcal{C}_{13}$, $\mathcal{C}_{15}$ are cycles of Lemma 5.1; $\mathcal{C}_{19} = (s_5, x_1, x_7, x_6, r_8)$, $\mathcal{C}_{20} = (s_5, c_{m-1}, d_1, b_1, r_8)$, $\mathcal{C}_{21} = (r_1, r_2, r_3, r_4, r_5)$, $\mathcal{C}_{22} = (r_3, r_4, r_5, r_6, r_7)$. $\square$

By using the above Lemmas we obtain the following proposition:

Proposition 5.4. *Let* G *be any graph from* $\mathcal{O} \cup \mathcal{F} \cup \mathcal{I}$*. Then every automorphism of* G *fixes* x_3.

Proof. Let α be an automorphism of G. The cycle $\mathcal{C}_6$ is the only 5-cycle in G touching the vertex x_3. By Property EP1 if $\alpha(x_3) = v$, the vertex v is a vertex contained just in one 5-cycle $\mathcal{C}$ of G. Thus, v is one of the vertices of Lemmas 5.1, 5.2 and 5.3 and $\alpha(\mathcal{C}_6) = \mathcal{C}$ where $\mathcal{C}$ is one of the 5-cycles highlighted in the proof of Lemmas 5.1, 5.2 and 5.3. We prove that $\alpha(x_3) \in \{x_1, x_3, x_5, x_6\}$. The assumption $\alpha(x_3) \notin \{x_1, x_3, x_5, x_6\}$ yields the following cases:

(a) $G \in \{O_2, F_2\}$ with $\alpha(x_3) = a_1$, then we obtain $\alpha(\mathcal{C}_6) = \mathcal{C}_{15}$ and $\alpha^{-1}(b_1) \in \{x_2, x_4\}$;

(b) $G \in \{O_m, F_m : m \geq 3\}$, with $\alpha(x_3) = a_1$, then we get $\alpha(\mathcal{C}_6) = \mathcal{C}_{13}$ and $\alpha^{-1}(b_1) \in \{x_2, x_4\}$;

(c) $G \in \{O_m, F_m : m \geq 2\}$ with $\alpha(x_3) = b_1$, then $\alpha^{-1}(a_1) \in \{x_2, x_4\}$;

(d) $G \in \{O_m : m \geq 1\}$ with $\alpha(x_3) = z_i$ with $i = 1$ or $i = 2$ or $i = 6$ or $i = 7$, then we obtain $\alpha^{-1}(z_j) \in \{x_2, x_4\}$ with $j = 2$ or $j = 1$ or $j = 7$ or $j = 6$, respectively;

(e) $G \in \{I_m : m \geq 1\}$ with $\alpha(x_3) = s_5$ or $\alpha(x_3) = r_1$ or $\alpha(x_3) = r_2$ or $\alpha(x_3) = r_6$ or $\alpha(x_3) = r_7$ or $\alpha(x_3) = r_8$, then we obtain $\alpha^{-1}(r_8) \in \{x_2, x_4\}$, or $\alpha^{-1}(r_2) \in \{x_2, x_4\}$, or $\alpha^{-1}(r_1) \in \{x_2, x_4\}$, or $\alpha^{-1}(r_7) \in \{x_2, x_4\}$, or $\alpha^{-1}(r_6) \in \{x_2, x_4\}$, or $\alpha^{-1}(s_5) \in \{x_2, x_4\}$, respectively;

(f) $G = I_2$ with $\alpha(x_3) = a_1$, then we get $\alpha(\mathcal{C}_6) = \mathcal{C}_{15}$ and $\alpha(x_9) \in \{d_1, x_6\}$;

(g) $G \in \{I_m : m \geq 3\}$ with $\alpha(x_3) = a_1$ implies $\alpha(\mathcal{C}_6) = \mathcal{C}_{13}$ and $\alpha(x_9) \in \{d_1, b_2\}$;

(h) $G \in \{O_m, F_m : m \geq 3\}$ with $\alpha(x_3) = c_{m-1}$, then we obtain $\alpha^{-1}(d_1) \in \{x_2, x_4\}$;

(i) $G \in \{O_2, F_2\}$ with $\alpha(x_3) = c_1$, then we get $\alpha(\mathcal{C}_6) = \mathcal{C}_{18}$ and $\alpha(x_9) \in \{x_6, x_7\}$.

We show that each one of these cases yields a contradiction.

Cases (a)–(e): While each of x_2 or x_4 is contained in precisely two 5-cycles (the cycles $\mathcal{C}_6$, $\mathcal{C}_7$ or $\mathcal{C}_5$, $\mathcal{C}_6$, respectively), the number of 5-cycles touching their image $\alpha(x_2), \alpha(x_4)$ is different from 2 (it is namely 1 by Lemmas 5.1, 5.2 and 5.3).

Case (f): Only two 5-cycles, $\mathcal{C}_6$ and $\mathcal{C}_7$, go through x_9 while the three cycles $\mathcal{C}_5$, $\mathcal{C}_8 = (x_6, d_1, b_1, a_1, x_5)$ and $\mathcal{C}_9 = (x_6, d_1, c_1, x_1, x_7)$ contain the vertex x_6 and the three cycles $\mathcal{C}_8, \mathcal{C}_9, \mathcal{C}_{20}$ touch the vertex d_1.

Case (g): Only two 5-cycles, $\mathcal{C}_6$ and $\mathcal{C}_7$, go through x_9 while the three cycles $\mathcal{C}_{13}, \mathcal{C}_{14}$ and $\mathcal{C}_{20}$, go through d_1 and the three cycles $\mathcal{C}_{13}, \mathcal{C}_{14}$ and $(b_2, d_2, b_3, a_3, a_2)$ contain the vertex b_2.

Case (h): There do not exist 6-cycles containing d_1 while the 6-cycle $(x_1, x_2, x_3, x_4, x_8, x_7)$ goes through x_2 or x_4.

Case (i): While x_9 is contained in precisely two 5-cycles, ($\mathcal{C}_6$ and $\mathcal{C}_7$), the three 5-cycles $\mathcal{C}_5$, $\mathcal{C}_{15}$ and $\mathcal{C}_{18}$ go through x_6 and the three 5-cycles $\mathcal{C}_5$, $\mathcal{C}_7, \mathcal{C}_{18}$ contain x_7.

Therefore, we have proved that $\alpha(x_3) \in \{x_1, x_3, x_5, x_6\}$. Hence, Proposition 4.2 implies that $\alpha(x_3) = x_3$ or $\alpha(x_3) = x_1$. The second case does not occur: if $\alpha \in \mathrm{Aut}(I_m)$, with $m \geq 1$, there is a different number of 5-cycles going through each vertex x_3 and x_1; if $\alpha \in \mathrm{Aut}(G)$, with $G \in \{O_m, F_m : m \geq 2\}$, there is a different number of 5-cycles containing each vertex x_3 and x_1; and finally, if $G \in \{O_1, F_1\}$, there is a different number of 8-cycles passing through each vertex x_3 and x_1 (see Lemmas 5.1 and 5.2). □

Proposition 5.5. *Let G be any graph from $\mathcal{O} \cup \mathcal{F} \cup \mathcal{I}$ with $\bar{G}_m$ as subgraph of G for an integer $m \geq 1$. Let α be an automorphism of G, then α fixes $\bar{G}_m$ setwise and the restriction α to $\bar{G}_m$ is either the involution $(x_{11}\ x_{17})$ $(x_{12}\ x_{16})(x_{13}\ x_{15})$ or the identity permutation.*

Proof. The statement follows from Proposition 5.4 and Corollary 4.5. □

Theorem 5.6. *Let G be any graph from $\mathcal{O} \cup \mathcal{F} \cup \mathcal{I}$, then the automorphism group of G is the trivial group.*

Proof. Let α be an automorphism of G with $G \in \mathcal{O} \cup \mathcal{F} \cup \mathcal{I}$. Define $\bar{V} = X$ if $m = 1$ and $\bar{V} = X \cup A_m \cup B_m \cup C_m \cup D_m$ if $m \geq 2$. Every vertex $v \in \bar{V}$ is adjacent to no more than one vertex $p_v \notin \bar{V}$. Let us consider the vertices x_5, x_6, x_1 if $m = 1$, or the vertices a_1, b_1, c_{m-1} if $m \geq 2$; Proposition 5.5 and Property EP3 imply that α fixes the vertices not belonging to $\bar{V}$ and adjacent to each of the vertices x_5, x_6, x_1 if $m = 1$, or vertices a_1, b_1, c_{m-1} if $m \geq 2$. In particular we get

- $\alpha(y_4) = y_4$, $\alpha(z_7) = z_7$ and $\alpha(z_1) = z_1$, if $G \in \mathcal{O}$;
- $\alpha(t_3) = t_3$, $\alpha(r_8) = r_8$ and $\alpha(r_1) = r_1$, if $G \in \mathcal{F}$;
- $\alpha(r_1) = r_1$, $\alpha(r_8) = r_8$ and $\alpha(s_5) = s_5$, if $G \in \mathcal{I}$. Moreover, in this case Property EP3 also implies that $\alpha(s_4) = s_4$.

By Proposition 5.5 we have only two cases:
I) The automorphism α acts on $\bar{G}_m$ as the permutation $(x_{11}\ x_{17})(x_{12}\ x_{16})$ $(x_{13}\ x_{15})$.

If $G \in \mathcal{O}$, then the pair $\alpha([x_{12}, y_3]) = [\alpha(x_{12}), \alpha(y_3)] = [x_{16}, \alpha(y_3)]$ is an edge and so the vertex $\alpha(y_3)$ is adjacent to x_{16}; Proposition 5.5 implies that $\alpha(X) = X$; thus $\alpha(y_3) = y_1$, hence $\alpha([y_3, y_4]) = [\alpha(y_3), \alpha(y_4)] = [y_1, y_4]$. A contradiction since $[y_1, y_4]$ is not an edge.

If $G \in \mathcal{F}$, then the pair $\alpha([x_{12}, t_2]) = [\alpha(x_{12}), \alpha(t_2)] = [x_{16}, \alpha(t_2)]$ is an edge with the vertex $\alpha(t_2)$ adjacent to x_{16}. From Proposition 5.5 we obtain

$\alpha(X) = X$; hence $\alpha(t_2) = t_1$, thus $\alpha([t_2, t_3]) = [\alpha(t_2), \alpha(t_3)] = [t_1, t_3]$. A contradiction since $[t_1, t_3]$ is not an edge.

If $G \in \mathcal{I}$, then the pair $\alpha([x_{12}, s_3]) = [\alpha(x_{12}), \alpha(s_3)] = [x_{16}, \alpha(s_3)]$ is an edge with the vertex $\alpha(s_3)$ adjacent to x_{16}; From Proposition 5.5 we get $\alpha(X) = X$, thus $\alpha(s_3) = s_1$, therefore $\alpha([s_3, s_4]) = [\alpha(s_3), \alpha(s_4)] = [s_1, s_4]$. A contradiction since $[s_1, s_4]$ is not an edge.

Therefore, this first case does not occur.

II) The automorphism α acts on $\bar{G}_m$ as the trivial permutation.

Let $G \in \mathcal{O}$. By Property EP3 we have $\alpha(y_3) = y_3$ (the vertices x_{12}, x_{11}, x_{13} are fixed by α); Property EP3 implies $\alpha(y_2) = y_2$ and $\alpha(y_1) = y_1$ (the vertices y_3, y_4, x_{12} and x_{16}, x_{17}, x_{15} are respectively fixed by α). By Property EP2 we also have $\alpha(z_2) = z_2$ (the vertices z_1, y_1 are fixed), $\alpha(z_3) = z_3$ (the vertices z_2, z_7 are fixed), $\alpha(z_4) = z_4$ (the vertices z_3, y_4 are fixed), $\alpha(z_5) = z_5$ (the vertices z_1, z_4 are fixed) and $\alpha(z_6) = z_6$ (the vertices z_5, z_7 are fixed). Therefore, α is the identity permutation on G.

Let $G \in \mathcal{F}$. By Property EP3 we obtain $\alpha(t_1) = t_1$ (the vertices x_{15}, x_{16}, x_{17} are fixed by α), $\alpha(t_2) = t_2$ (the vertices x_{12}, x_{11}, x_{13} are fixed by α), thus $\alpha(r_3) = r_3$ (the vertices x_{16}, t_1, t_2 are fixed) and $\alpha(r_6) = r_6$ (the vertices a_1, t_3, t_2 or x_5, t_3, t_2 if $m = 1$, are fixed). By Property EP2 we also have $\alpha(r_2) = r_2$ (the vertices r_1, r_3 are fixed), $\alpha(r_4) = r_4$ (the vertices r_3, r_8 are fixed), $\alpha(r_5) = r_5$ (the vertices r_4, r_6 are fixed) and $\alpha(r_7) = r_7$ (the vertices r_6, r_8 are fixed). Therefore, α is the identity permutation on G.

Let $G \in \mathcal{I}$. By Property EP3 we get $\alpha(s_3) = s_3$ (the vertices x_{12}, x_{11}, x_{13} are fixed by α), hence $\alpha(s_2) = s_2$ (the vertices x_{12}, s_3, s_4 are fixed). By Property EP2 we obtain $\alpha(s_1) = s_1$ (the vertices s_2 and x_{16} are fixed). By Property EP3 we have $\alpha(r_2) = r_2$ (the vertices s_1, s_2, x_{16} are fixed), $\alpha(r_4) = r_4$ (the vertices s_3, s_4, s_5 are fixed) and $\alpha(r_6) = r_6$ (the vertices s_1, s_2, s_3 are fixed). Therefore, by Property EP2 the vertices r_3, r_5 and r_7 are also fixed. The automorphism α is the identity permutation on G. The statement follows. □

Corollary 5.7. *For every possible order greater than 26 there exists an asymmetric nearly hamiltonian snark of that order.*

Proof. The nearly hamiltonian snarks O and F shown in Figure 3 have order 24 while the nearly hamiltonian snark I shown in Figure 4 has order 26. From Corollary 2.2 the nearly hamiltonian snarks O_m and F_m have order $24 + 4m$ while I_m has order $26 + 4m$. The statement follows from Theorem 5.6. □

References

[1] G. Brinkmann, *Fast generation of cubic graphs*, J. Graph Theory 23 (1996), 139-149.

[2] G. Brinkmann, E. Steffen, *Snarks and reducibility*, Ars Combin. 50 (1998), 292-296.

[3] P.J. Cameron, *Automorphisms of graphs*, in "Topics in Algebraic Graph Theory" , Encyclopedia of Mathematics and its Applications, 102, Cambridge University Press, Cambridge, 137–155, 2004.

[4] C.Y. Chao, C.F. Dong, *On a class of identity graphs*, Graph theory and its applications: East and West (Jinan, 1986), 97–106, Ann. New York Acad. Sci., 576, New York Acad. Sci., New York, 1989.

[5] A. Cavicchioli, M. Meschiari, B. Ruini, F. Spaggiari, *A Survey on snarks and new results: products, reducibility and a computer research*, J. Graph Theory 28 (1998), 57-95.

[6] A. Cavicchioli, T.E. Murgolo, B. Ruini, F. Spaggiari, *Special classes of snarks*, Acta Appl. Math. 76 (2003), 57–88.

[7] U.A. Celmins, E.R. Swart, *The construction of snarks*, Dept. of Combinatorics and Optimizations, Univ. of Waterloo, Research Report CORR (1979), 79-18.

[8] A.G. Chetwynd, R.J. Wilson, *Snarks and supersnarks*, The theory and applications of graphs, (Kalamazoo, Mich. 1980), Wiley, New york, 1981, pp. 215-241.

[9] P. Erdős, A. Rényi, *Asymmetric graphs*, Acta Math. Acad. Sci. Hungar. 14 (1963), 295–315.

[10] P. Erdős, R.J. Wilson, *On the chromatic index af almost all graphs*, J. Combinatorial Theory B 23 (1977), 255-257.

[11] C. Fiori, B. Ruini, *Infinite classes of dihedral snarks*, Mediterr. J. Math. 5 (2008), 199-210.

[12] S. Fiorini, R.J. Wilson, *Edge-colouring of graphs*, Research Notes in Mathematics 16, Pitman, London, 1977.

[13] M. Gardner, *Mathematical Games: Snarks, Boojums and other conjectures related to the four-color-map theorem*, Scientific American 234 (:4) (April 1976), 126-130.

[14] M.K. Goldberg, *Construction of class 2 graphs with maximum vertex degree 3*, J. Combin. Theory Ser. B 31 (1981), 282-291.

[15] R. Isaacs, *Infinite families of non trivial graphs which are not Tait colorable*, Amer. Math. Monthly 82 (1975), 221-239.

[16] T.R. Jensen, B. Totf, *Graph Coloring Problems*, Interscience Series in Discrete Mathematics and Optimization, Wiley, New York, 1995.

[17] H.A. Kierstead, *Asymmetric graph coloring games*, J. Graph Theory 48 (2005), 169–185.

[18] R. Nedela, M. Škoviera, *Decompositions and reductions of snarks*, J. Graph Theory 22 (3) (1996), 253-279.

[19] N. Punnim, S. Thaithae, *The Hamiltonian number of cubic graphs*, Lecture Notes Comput. Sci. Springer, 4535 (2008), 213-223.

[20] N. Punnim, V. Saenpholphat, S. Thaithae, *Almost Hamiltonian cubic graphs*, Inter. J. Computer Sci. Network Security, 7 (2007), 83-86.

[21] R.C. Ronald, R.J. Wilson *An atlas of graphs*, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998.

[22] E. Steffen, *Classifications and characterizations of snarks*, Discrete Math. 188 (1998), 183-203.

[23] V.G. Vizing, *On an estimate of the chromatic class of a p-graph*, Diskret. Analiz. 3 (1964), 25-30.

[24] J.J. Watkins, R.J. Wilson, *A survey of snarks*, Graph Theory, Combinatorics, and Applications, 2 (Kalamazoo, MI, 1988) Wiley-Interscience, New York, 1991, 1129-1144.

[25] H.A. Jung, S. Kyaw, B. Wei, *Almost-Hamiltonian graphs*, Contemporary methods in graph theory, 409-427, Bibliographisches Inst., Mannheim, 1990.

[26] E.W. Weisstein, *Graph Automorphism.* From MathWord–A Wolfram Web Resource. (http://mathworld.wolfram.com/GraphAutomor phism.html).

[27] E.W. Weisstein, *Identity Graph.* From MathWord–A Wolfram Web Resource. (http://mathworld.wolfram.com/IdentityGraph.html).

Authors' addresses:

Carla Fiori, Dipartimento di Matematica, Università di Modena e Reggio Emilia, via Campi 213/B, 41100 Modena (Italy).
E–mail: `carla.fiori@unimore.it`

Beatrice Ruini, Dipartimento di Matematica, Università di Modena e Reggio Emilia, via Campi 213/B, 41100 Modena (Italy).
E–mail: `beatrice.ruini@unimore.it`